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# TRANSVERSE VIBRATIONS OF COMPOSITE, CIRCULAR ANNULAR MEMBRANES: EXACT SOLUTION

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## 1. Introduction

Structural elements with non-uniform material properties are technically important in many engineering situations. On the other hand, certain manufacturing processes may generate undesired variations in the mechanical properties of a system and, in turn, these variations will affect the static and dynamic response of the structural element.

The dynamic analysis of membranes with discontinuously varying material properties has been the subject of many recent investigations. In a well-known paper, Spence and Horgan [1] found upper and lower bounds for the natural frequencies of vibration of a circular membrane with stepped radial density and they showed that eigenvalue estimation techniques based on an integral equation approach are more effective than classical variational techniques. A conformal mapping approach was used in reference [2] in the case of composite membranes of regular polygonal shape whose inner circular core possesses a density $\rho_1$, while the remaining is characterized by $\rho_0$.

In general previous investigations deal with composite, simply connected membranes.

Vibrating circular annular membranes of continuous and discontinuous variation of the density in a radial direction have been studied recently by means of an approximate variational approach [3]. An independent solution has also been obtained using the differential quadrature method when the density varies in a continuous fashion, the eigenvalues being in good agreement with those obtained previously [3].

The present paper deals with an exact solution of the problem of transverse vibrations of a composite, doubly connected membrane with discontinuously varying thickness, Figure 1.

## 2. Solution of the problem

Making use of the classical theory of vibrating membranes, the problem is governed by

$$\nabla^2 u_i = \frac{1}{\alpha_i^2}\frac{\partial^2 u_i}{\partial t^2}, \qquad i = 1, 2, \tag{1}$$

where $\alpha_i = \sqrt{S/\rho_i}$.

Now using the method of separation of variables, one writes

$$u_i = U_i(r)\, e^{in\theta}\, e^{i\omega t}, \qquad n = 0, 1, 2, \ldots, \tag{2}$$

and substituting in equation (1) one obtains

$$\frac{d^2 U_i}{dr^2} + \frac{1}{r}\frac{dU_i}{dr} + \left(\frac{\omega^2}{\alpha_i^2} - \frac{n^2}{r^2}\right)U_i = 0, \tag{3}$$

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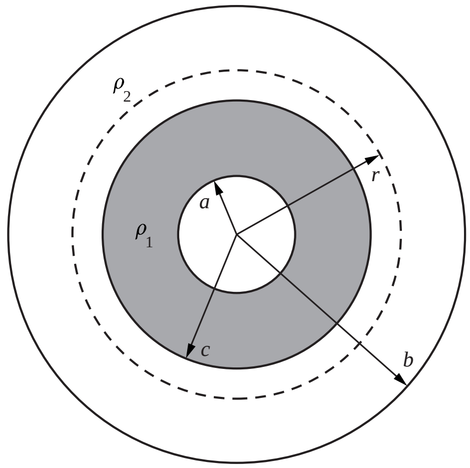


Figure 1. Non-homogeneous membrane executing transverse vibrations.

its solution being

$$U_1 = A_n J_n\left(\frac{\omega}{\alpha_1} r\right) + B_n Y_n\left(\frac{\omega}{\alpha_1} r\right), \qquad U_2 = C_n J_n\left(\frac{\omega}{\alpha_2} r\right) + D_n Y_n\left(\frac{\omega}{\alpha_2} r\right). \qquad (4a, b)$$

The frequency determinant is generated using the boundary conditions at $r = a, b$:

$$U_1(a) = U_2(b) = 0 \qquad (5a, b)$$

and the compatibility (or continuity) conditions at $r = c$:

$$U_1(c) = U_2(c), \qquad \frac{dU_1}{dr}(c) = \frac{dU_2}{dr}(c), \qquad (6, 7)$$

The present study is concerned with the determination of frequency coefficients corresponding to axisymmetric modes of vibration ($n = 0$). Accordingly, expressions (4) become

$$U_1 = A_0 J_0\left(\frac{\omega}{\alpha_1} r\right) + B_0 Y_0\left(\frac{\omega}{\alpha_1} r\right), \qquad (8a)$$

$$U_2 = C_0 J_0\left(\frac{\omega}{\alpha_2} r\right) + D_0 Y_0\left(\frac{\omega}{\alpha_2} r\right). \qquad (8b)$$

Substituting equations (8) into equations (5), (6) and (7), one obtains

$$A_0 J_0\left(\frac{\omega}{\alpha_1} a\right) + B_0 Y_0\left(\frac{\omega}{\alpha_1} a\right) = 0,$$

$$A_0 J_0\left(\frac{\omega}{\alpha_1} c\right) + B_0 Y_0\left(\frac{\omega}{\alpha_1} c\right) - C_0 J_0\left(\frac{\omega}{\alpha_2} c\right) - D_0 Y_0\left(\frac{\omega}{\alpha_2} c\right) = 0,$$

$$-A_0\sqrt{\rho_1}J_1\left(\frac{\omega}{\alpha_1}c\right)-B_0\sqrt{\rho_1}Y_1\left(\frac{\omega}{\alpha_1}c\right)+C_0\sqrt{\rho_2}J_1\left(\frac{\omega}{\alpha_2}c\right)+D_0\sqrt{\rho_2}Y_1\left(\frac{\omega}{\alpha_2}c\right)=0,$$

$$C_0J_0\left(\frac{\omega}{\alpha_2}b\right)+D_0Y_0\left(\frac{\omega}{\alpha_2}b\right)=0. \qquad (9)$$

TABLE 1

*Values of* $\Omega_{0i}=\sqrt{\rho_2/S}\omega_{0i}b$ *for a composite, circular annular membrane* $(a/b=0{\cdot}1;\ c/b=0{\cdot}5)$

| $\rho_2/\rho_1$ | $\Omega_{01}$ | $\Omega_{02}$ | $\Omega_{03}$ | $\Omega_{04}$ | $\Omega_{05}$ |
|---|---|---|---|---|---|
| 0·10 | 3·9875 | 9·5624 | 15·3610 | 20·7426 | 24·6925 |
| 0·50 | 3·6790 | 8·0916 | 11·7855 | 15·9000 | 20·1864 |
| 0·90 | 3·3830 | 7·0370 | 10·5903 | 14·2391 | 17·7684 |
| 1·50 | 3·0050 | 6·2396 | 9·4429 | 12·6209 | 15·8262 |
| 2·00 | 2·7541 | 5·8671 | 8·6760 | 11·8511 | 14·5560 |
| 5·00 | 1·9324 | 4·6959 | 6·5718 | 8·8662 | 11·5082 |
| 10·00 | 1·4127 | 3·6006 | 5·5779 | 6·8254 | 8·7498 |

TABLE 2

*Values of* $\Omega_{0i}=\sqrt{\rho_2/S}\omega_{0i}b$ *for a composite, circular annular membrane* $(a/b=0{\cdot}20;\ c/b=0{\cdot}5)$

| $\rho_2/\rho_1$ | $\Omega_{01}$ | $\Omega_{02}$ | $\Omega_{03}$ | $\Omega_{04}$ | $\Omega_{05}$ |
|---|---|---|---|---|---|
| 0·10 | 4·2795 | 9·8238 | 15·7129 | 21·6171 | 27·2512 |
| 0·50 | 4·0763 | 8·8789 | 13·3222 | 17·3649 | 22·0097 |
| 0·90 | 3·8677 | 7·9706 | 11·9430 | 15·9677 | 20·0190 |
| 1·50 | 3·5672 | 7·0907 | 10·9376 | 14·3434 | 18·1608 |
| 2·00 | 3·3420 | 6·6561 | 10·2865 | 13·4090 | 17·1499 |
| 5·00 | 2·4665 | 5·5944 | 7·6971 | 11·0269 | 13·1399 |
| 10·00 | 1·8346 | 4·6245 | 6·3828 | 8·4369 | 11·1849 |

TABLE 3

*Values of* $\Omega_{0i}=\sqrt{\rho_2/S}\omega_{0i}b$ *for a composite, circular annular membrane* $(a/b=0{\cdot}30;\ c/b=0{\cdot}5)$

| $\rho_2/\rho_1$ | $\Omega_{01}$ | $\Omega_{02}$ | $\Omega_{03}$ | $\Omega_{04}$ | $\Omega_{05}$ |
|---|---|---|---|---|---|
| 0·10 | 4·6618 | 10·1788 | 16·0664 | 22·0801 | 28·1212 |
| 0·50 | 4·5549 | 9·6635 | 14·8762 | 19·7549 | 24·2906 |
| 0·90 | 4·4415 | 9·0777 | 13·6675 | 18·1727 | 22·7286 |
| 1·50 | 4·2636 | 8·2792 | 12·6145 | 16·9740 | 20·9602 |
| 2·00 | 4·1129 | 7·7717 | 12·1207 | 16·1096 | 19·8476 |
| 5·00 | 3·3295 | 6·4659 | 10·1371 | 12·9922 | 16·8898 |
| 10·00 | 2·5747 | 5·8447 | 7·8954 | 11·5638 | 13·3880 |

Introducing the dimensionless parameters $r_1 = a/b$, $r_2 = c/b$, $\rho = \rho_1/\rho_2$ and $\Omega = \sqrt{\rho_2/S}\,\omega b$, the secular determinant results:

$$\begin{vmatrix} J_0(\sqrt{\rho}r_1\Omega) & Y_0(\sqrt{\rho}r_1\Omega) & 0 & 0 \\ J_0(\sqrt{\rho}r_2\Omega) & Y_0(\sqrt{\rho}r_2\Omega) & -J_0(r_2\Omega) & -Y_0(r_2\Omega) \\ -\sqrt{\rho}J_1(\sqrt{\rho}r_2\Omega) & -\sqrt{\rho}Y_1(\sqrt{\rho}r_2\Omega) & J_1(r_2\Omega) & Y_1(r_2\Omega) \\ 0 & 0 & J_0(\Omega) & Y_0(\Omega) \end{vmatrix} = 0. \qquad (10)$$

## 3. NUMERICAL RESULTS

The determination of the first five roots of the determinantal equation (10) has been greatly facilitated by the use of *Mathematica* [4].

Tables 1–3 present values of $\Omega_{0i}$ for $a/b = 0{\cdot}1$, $0{\cdot}2$ and $0{\cdot}3$, respectively, while $c/b = 0{\cdot}5$ for the three configurations.

It is important to point out that the fundamental eigenvalues computed by a variational approach in reference [3] have been verified using the present, exact solution. Since they are upper bounds they are higher than the exact eigenvalues, the differences being, in general, less than 1%.

## ACKNOWLEDGMENTS

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